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A BOUNDARY VALUE PROBLEM FOR AN ELLIPTIC EQUATION WITH ASYMMETRIC COEFFICIENTS IN A NONSCHLICHT DOMAIN

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Abstract: We propose some minimum principle for the quadratic energy functional of an elliptic boundary value problem describing a transport process with asymmetric tensor coefficients in a nonschlicht domain. We prove the existence and uniqueness of a weak solution in the energy space. The energy norm equals the entropy production rate.

Keywords: energy functional, elliptic equation, non-self-adjoint operator, nonschlicht domain

1. Introduction. The operators are not self-adjoint of the elliptic boundary value problems traditional to description of transport processes in gyrotropic media. Therefore, we cannot apply to them the energy methods of [1] enabling the construction of effective approximate and numeric algorithms. Starting with [2], the author poses and studies some new boundary value problems that describe the same physical processes but involve positive symmetric operators. For various two-dimensional boundary value problems the author has justified the minimum principle for the corresponding quadratic energy functional. He has also constructed a difference-variational scheme and demonstrated the effectiveness of the multigrid method [3]. The three-dimensional problem is considered in [4].

A typical example of a gyrotropic conductor is a partially ionized plasma in a magnetic field. Mathematical modeling of electric fields in Earth's ionosphere leads to a two-dimensional problem in a nonschlicht domain [5]. Under some additional assumptions [5], it reduces to a problem in a schlicht simply connected domain. This problem is restated in [6] as an elliptic boundary value problem with a symmetric operator for which the minimum principle is justified for the quadratic energy functional.

The aim of the present article is to extend the energy method to the problem in a nonschlicht domain.

We give proofs only in the case of a boundary condition on the exterior boundary which corresponds to extraction of a near-boundary singularity. In the last section we merely list changes in the statements and proofs for simpler problems.

2. Statement of a boundary value problem. In the mathematical modeling of large-scale electric fields, we usually consider Earth's ionosphere as a two-dimensional conductor and solve a quasistationary electrical conduction problem [5]. After some transformations, there appears a problem in a domain Ω composed of three planar subdomains Ω^E, Ω^N, and Ω^S. In the simplified model in which the geomagnetic field is assumed to be dipole, Ω^N and Ω^S are disks of radius r_0 and Ω^E is a ring $r_0 < r < 1$. In this case all three subdomains are glued together along a circle of radius r_0. In the general case the subdomains are not glued to one another in the geometric sense, although the corresponding points on the boundaries of the three subdomains are interrelated by the equipotentiality condition. This circumstance slightly complicates the statement of the conjugation conditions below as compared with the above-indicated particular case.

In order to use the Sobolev embedding theorem, we suppose that each of the subdomains Ω^E, Ω^N, and Ω^S is a connected union of finitely many domains each of which is starlike with respect to some disk. The subdomains Ω^N and Ω^S are supposed to be simply connected. They are bounded by closed smooth curves Γ^N and Γ^S. The subdomain Ω^E is homeomorphic to a ring. Its interior boundary Γ^E and the exterior boundary Γ are closed smooth curves as well.

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We use the arclength l as a coordinate on each boundary and let the indices n and l mark the normal and tangential components of vectors. There is a smooth one-to-one correspondence between the points on Γ^E, Γ^N, and Γ^S which is given by the functions $l^N(l^E)$ and $l^S(l^E)$. Without loss of generality we may assume that $l^N(0) = l^S(0) = 0$. It is convenient to write down the constraints on the functions as follows:

$$0 < c_1 \leq \frac{dl^N}{dl^E} \leq c_2 < \infty, \quad 0 < c_1 \leq \frac{dl^S}{dl^E} \leq c_2 < \infty. \tag{1}$$

The outward normals are regarded as positive, while the positive direction of a tangent keeps a domain on the left, except for Γ^E on which we choose the opposite direction of l so that the directions on Γ^N and Γ^S agree. Otherwise the functions in (1) would have the opposite sign.

The current density $\mathbf{J}$ and the electrical field strength $\mathbf{E}$ in each subdomain satisfy the law of conservation of charge and the Faradey law of electro-magnetic induction:

$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} = Q(x,y), \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = G(x,y), \tag{2}$$

where $Q(x,y)$ and $G(x,y)$ are given functions; usually $G = 0$.

The components of $\mathbf{E}$ and $\mathbf{J}$ are connected by the Ohm law

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{3}$$

The components of the conductivity tensor σ are given functions of coordinates. By the Hall phenomenon, the tensor σ is asymmetric.

The boundary condition on the exterior boundary Γ has the form

$$\left(J_n - \frac{\partial}{\partial l}(A(l)E_l) \right)\bigg|_\Gamma = 0, \tag{4}$$

where $A(l)$ is a given function.

This boundary condition corresponds to extraction of a narrow near-boundary strip of high conductivity, and for $A = 0$ becomes the condition on an ideal isolator.

The corresponding points on the boundaries Γ^E, Γ^N, and Γ^S of the three subdomains Ω^E, Ω^N, and Ω^S are interrelated by the equipotentiality condition. Therefore, from the law of conservation of charge and the induction equation we obtain the following conjugation conditions [5]:

$$J_n(l^E) + \frac{dl^N}{dl^E} J_n(l^N) + \frac{dl^S}{dl^E} J_n(l^S) = 0, \tag{5}$$

$$E_l(l^E) - \frac{dl^N}{dl^E} E_l(l^N) = 0, \quad E_l(l^E) - \frac{dl^S}{dl^E} E_l(l^S) = 0. \tag{6}$$

Here the boundary values of the components of vectors are the boundary values in the subdomain with the same superscript E, N, or S as that of the coordinate l; moreover, the correspondence between the points Γ^E, Γ^N, and Γ^S is given by the functions $l^N(l^E)$ and $l^S(l^E)$.

In the particular case when the subdomains are glued together geometrically, we can introduce the unified arclength $l^N = l^E = l^S$ on the boundaries Γ^E, Γ^N, and Γ^S, and hence all derivatives in (5) and (6) are equal to 1. Therefore, condition (5) amounts to the vanishing of the sum of the current densities of the three subdomains on the common boundary and condition (6) becomes the continuity condition for the tangential component of the electric field strength upon passage through the boundary.

For solvability of the boundary value problem (2)–(6), the right-hand side must satisfy the conditions

$$\iint_\Omega Q\,dxdy = 0, \quad \iint_{\Omega^N} G\,dxdy = \iint_{\Omega^S} G\,dxdy \tag{7}$$

which result from integration of the first equation in (2) over Ω and the second equation over Ω^N and Ω^S in view of homogeneity of the boundary condition (4) on the exterior boundary and the conjugation conditions (5) and (6) on the interior boundary. Below we do not indicate the domain of integration if this is the whole domain Ω.

3. Uniqueness of a solution. In the next sections we state a new problem whose solution is a solution to the original problem (2)–(6). Existence of a solution to the new problem implies existence of a solution to the original problem, whereas uniqueness must be proven independently.

We state the necessary constraints on the distribution of σ inside the domain and on the integral conductivity distribution in the near-boundary strip predetermined by the boundary condition (4).

The conductivity tensor σ is supposed to be uniformly bounded in Ω and its symmetric part is assumed uniformly positive definite. It is convenient to write down these properties as follows: There exist nonzero finite positive numbers c_3 and c_4 such that

$$\lambda \geq c_3, \quad \det(\sigma)/\lambda \leq c_4 \tag{8}$$

for all points of the domain, where $\det(\sigma)$ is the determinant of the matrix σ and λ is the least eigenvalue of the symmetric matrix $(\sigma + \sigma^T)/2$. In the practically important particular case when the conductor is gyrotropic the functions subject to (8) are the Pedersen and Cowling conductivities. It is easy to validate the relations

$$\sigma_{xx} \geq \lambda, \quad \sigma_{yy} \geq \lambda$$

as well as uniform boundedness of all entries of σ, provided that (8) is satisfied.

We suppose that $A(l)$ in the boundary condition (4) determining a near-boundary strip satisfies the conditions

$$0 < c_5 \leq A(l) \leq c_6 < \infty. \tag{9}$$

Lemma 1. *Suppose that the coefficients σ and $A(l)$ satisfy* (8) *and* (9) *in a three-sheeted domain Ω whose subdomains Ω^E, Ω^N, and Ω^S are bounded by piecewise smooth curves. Then problem* (2)–(6) *has at most one smooth solution.*

Suppose the contrary; i.e., suppose that there are two solutions. Then their differences $\mathbf{E}$ and $\mathbf{J}$ satisfy (2)–(6) with the zero right-hand sides $Q = 0$ and $G = 0$.

In the proof we use the function V:

$$\mathbf{E} = -\operatorname{grad} V. \tag{10}$$

This function exists in each subdomain, since the vector field $\mathbf{E}$ is irrotational by the second equation of (2).

Since the adding a constant to V does not change $\operatorname{grad} V$, we may assume that $V = 0$ at the point $l^E = 0$. Similarly, we add a constant to V in Ω^N and Ω^S so that $V = 0$ on Γ^N and Γ^S at the points $l^N = 0$ and $l^S = 0$.

By (10), the function V on Γ^E is given by the integral

$$V(l^E) = -\int_0^{l^E} E_l(\tilde{l})\, d\tilde{l}.$$

It follows from (6) that

$$-\int_0^{l^N} E_l(\tilde{l}^N)\, d\tilde{l}^N = -\int_0^{l^E} E_l(\tilde{l}^E)\, d\tilde{l}^E$$

and hence

$$V(l^N) = V(l^E). \tag{11}$$

Similarly,

$$V(l^S) = V(l^E); \tag{12}$$

i.e., V is continuous in the whole domain Ω.

By assumption $\mathbf{E}$ is not identically zero; therefore, we can find a number $\varepsilon > 0$ and a point such that $|\mathbf{E}| = 2\varepsilon$. In view of smoothness of $\mathbf{E}$, we can choose a neighborhood of this point in which $|\mathbf{E}| > \varepsilon$. Denote the area of this neighborhood by x_0^2.

Consider the following integral over the whole domain Ω:

$$w = \iint \mathbf{J}^T \mathbf{E}\, dxdy.$$

Using (10), we can rewrite the integral as

$$w = \iint (\operatorname{grad} V)^T \sigma \operatorname{grad} V\, dxdy. \tag{13}$$

We can replace σ in this integral with its symmetric part, since we in fact calculate the quadratic form

$$w = \iint (\operatorname{grad} V)^T \frac{\sigma + \sigma^T}{2} \operatorname{grad} V\, dxdy.$$

By positive definiteness of $(\sigma + \sigma^T)/2$, (8), the integrand is nonnegative. Therefore, the integral over the whole domain Ω is not less than the integral over the selected neighborhood; hence,

$$w \geq x_0^2 \varepsilon^2 c_3 > 0. \tag{14}$$

Now, we turn to the integral written down in the shape (13). Transform identically the integrand:

$$w = \iint (-V \operatorname{div}(\sigma \operatorname{grad} V) + \operatorname{div}(V \sigma \operatorname{grad} V))\, dxdy.$$

The first term equals $V \operatorname{div}(\sigma \mathbf{E}) = V \operatorname{div} \mathbf{J} = 0$, since the first equation of (2) holds with the zero right-hand side. Transform the remaining integral by using the Gauss–Ostrogradskiĭ theorem in each of the subdomains:

$$w = \oint\limits_{\Gamma} (\sigma \operatorname{grad} V)_n\, dl + \oint\limits_{\Gamma^E} V(\sigma \operatorname{grad} V)_n\, dl^E + \oint\limits_{\Gamma^N} V(\operatorname{grad} V)_n\, dl^N + \oint\limits_{\Gamma^S} V(\sigma \operatorname{grad} V)_n\, dl^S.$$

The sign of the second integral remains positive, despite dl^E has the opposite sign, since the direction of traversing Γ^E has changed. Returning to (10) and (3), we obtain

$$w = -\oint\limits_{\Gamma} V J_n\, dl - \oint\limits_{\Gamma^E} V J_n\, dl^E - \oint\limits_{\Gamma^N} V J_n\, dl^N - \oint\limits_{\Gamma^S} V J_n\, dl^S.$$

By (11) and (12), the integrands V in the last three integrals are the same; so the sum of the integrals equals

$$\oint\limits_{\Gamma^E} V \left\{ J_n(l^E) + \frac{dl^N}{dl^E} J_n(l^N) + \frac{dl^S}{dl^E} J_n(l^S) \right\} dl^E.$$

Here we simultaneously changed the signs of dl^N and dl^S and the directions of traversing Γ^N and Γ^S. The sum in braces vanishes by (5). We have

$$w = -\oint\limits_{\Gamma} V J_n\, dl. \tag{15}$$

By the boundary condition (4), we can transform this integral as follows:

$$w = -\oint\limits_\Gamma V \frac{\partial}{\partial l}\big(A(l)E_l\big)\, dl.$$

We transform identically the integrand:

$$w = \oint\limits_\Gamma \left(-\frac{\partial}{\partial l}(VA(l)E_l) + \frac{\partial V}{\partial l} A(l)E_l\right) dl.$$

The integral of the first term vanishes, since integration is carried over a closed contour. We eliminate V in the remaining integral by means of (10):

$$w = -\oint\limits_\Gamma A(l)E_l^2\, dl.$$

From strict positivity of $A(l)$, (9), we obtain

$$w \le -c_5 \oint\limits_\Gamma E_l^2\, dl \le 0$$

which contradicts (14); hence, the assumption that $\mathbf{E}$ is different from the identical zero is false. Thereby Lemma 1 is proven.

4. Symmetrization of the boundary value problem. If, as it is usual for $G \equiv 0$, we pass from (2) and (3) to an equation in the electric potential V of (10),

$$-\frac{\partial}{\partial x}\left(\sigma_{xx}\frac{\partial V}{\partial x} + \sigma_{xy}\frac{\partial V}{\partial y}\right) - \frac{\partial}{\partial y}\left(\sigma_{yx}\frac{\partial V}{\partial x} + \sigma_{yy}\frac{\partial V}{\partial y}\right) = Q,$$

or to an equation in the stream function for $Q \equiv 0$ then, owing to $\sigma_{xy} \ne \sigma_{yx}$, we come to boundary value problems with non-self-adjoint operators.

We obtain self-adjoint operators if, according to [3], we introduce a pair F, P of potentials such that

$$\mathbf{E} = -S\sigma^T \operatorname{grad} F + \operatorname{S rot} P, \tag{16}$$

where the matrix S is defined by the symmetric part of σ,

$$2S^{-1} = \sigma + \sigma^T, \tag{17}$$

and the vector $\operatorname{rot} P = (\partial P/\partial y, -\partial P/\partial x)$ comprises only the x- and y-components of the vorticity of the vector function which has only the z-component P.

Omitting the heuristic arguments which are similar to [3], we consider some quadratic functional and prove that minimizing it corresponds to solving the original problem (2)–(6). In Section 8 we show why this functional is called the energy functional. The statement of the problem with a self-adjoint operator arises as the minimization condition for the energy functional. The shape of equations and boundary conditions for the new functions F and P corresponds to substitution of (16) for $\mathbf{E}$ in the original problem (2)–(6) supplemented with the so-called principal boundary conditions (19)–(22) for F and P which are in a sense adjoint to (4)–(6). In particular, equations (2) and (3) inside the domain take the form

$$\operatorname{div}\big(-\sigma S\sigma^T \operatorname{grad} F + \sigma \operatorname{S rot} P\big) = Q, \quad \operatorname{rot}_z\big(-S\sigma^T \operatorname{grad} F + \operatorname{S rot} P\big) = G. \tag{18}$$

5. The energy space. We consider a pair F, P of smooth functions satisfying the following so-called principal conditions which are in a sense adjoint to (4)–(6). On the exterior boundary Γ we have

$$F(l) - \int\limits_0^l \frac{P(\tilde{l})}{A(\tilde{l})} d\tilde{l} = 0, \quad \oint\limits_\Gamma F(l)\, dl = 0, \quad \oint\limits_\Gamma \frac{P(l)}{A(l)}\, dl = 0. \tag{19}$$

We require F to be continuous on the interior boundaries:

$$F(l^E) = F(l^N) = F(l^S), \tag{20}$$

while P is subject to a condition similar to the summation condition for the stream functions:

$$P(l^E) = P(l^N) + P(l^S). \tag{21}$$

One more condition means the vanishing of the mean of $P(l^N)$ but after been mapped to Γ^E rather than considered on the curve Γ^N itself. This condition eliminates an arbitrary additive constant to within which the function P is actually determined in Ω^N:

$$\oint\limits_{\Gamma^N} P(l^N) \frac{dl^E}{dl^N}\, dl^N = 0. \tag{22}$$

We define the energy inner product as

$$\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix}\right] = \iint \begin{pmatrix} \operatorname{grad} u \\ \operatorname{rot} v \end{pmatrix}^T \begin{pmatrix} \sigma s \sigma^T & -\sigma s \\ -s\sigma^T & s \end{pmatrix} \begin{pmatrix} \operatorname{grad} F \\ \operatorname{rot} P \end{pmatrix} dxdy. \tag{23}$$

This is a symmetric bilinear form. Prove its positive definiteness on smooth functions. First, consider the auxiliary integral

$$\iint (\operatorname{grad} F)^T \operatorname{rot} P\, dxdy. \tag{24}$$

Transform identically the integrand:

$$(\operatorname{grad} F)^T \operatorname{rot} P = -\operatorname{rot}_z(P\ \operatorname{grad} F) + P\ \operatorname{rot}_z(\operatorname{grad} F).$$

The second summand vanishes identically. Using the Gauss-Ostrogradskiĭ formula, we can transform the integrals of the first summand into boundary integrals, coming to

$$-\oint\limits_{\Gamma^N} P\frac{\partial F}{\partial l^N}\, dl^N - \oint\limits_{\Gamma^S} P\frac{\partial F}{\partial l^S}\, dl^S + \oint\limits_{\Gamma^E} P\frac{\partial F}{\partial l^E}\, dl^E - \oint\limits_{\Gamma} P\frac{\partial F}{\partial l}\, dl. \tag{25}$$

Gather the three integrals over interior boundaries by using (20):

$$\oint \frac{\partial F}{\partial l^E}(P(l^E) - P(l^N) - P(l^S))\, dl^E.$$

The integrand vanishes by (21). In (25) we are left only with the integral over the exterior boundary. We transform it by using the boundary conditions (19) which amount to the following equality on smooth functions:

$$\left.\left(P - A(l)\frac{\partial F}{\partial l}\right)\right|_\Gamma = 0. \tag{26}$$

We obtain

$$-\oint\limits_\Gamma \frac{P^2}{A(l)}\, dl. \tag{27}$$

By positivity of $A(l)$, (9), this integral is nonpositive. Therefore, so is the whole integral (24). Add the doubled integral (24) to the quadratic form (23). The matrix of the quadratic form of the integrand takes the shape

$$\begin{pmatrix} \sigma S \sigma^T & I - \sigma S \\ I - S\sigma^T & S \end{pmatrix}, \tag{28}$$

where I is the identity $(2*2)$-matrix.

It is easy to prove [3] that this symmetric matrix is positive definite and its four eigenvalues lie in the interval

$$\frac{1}{2}\sqrt{\frac{c_3}{c_4}}, \quad 2\sqrt{\frac{c_4}{c_3}}, \tag{29}$$

provided that S is defined by (17) and σ satisfies (8).

Therefore, the value of the modified quadratic form is estimated from below and above by

$$\iint \left\{ (\operatorname{grad} F)^2 + (\operatorname{rot} P)^2 \right\} dxdy$$

with the respective coefficients (29).

Since the value of the auxiliary integral (27) is nonpositive, for the original quadratic form we obtain

$$\left[\begin{pmatrix} F \\ P \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix} \right] \geq \frac{1}{2}\sqrt{\frac{c_3}{c_4}} \iint \{ (\operatorname{grad} F)^2 + (\operatorname{rot} P)^2 \} \, dxdy, \tag{30}$$

$$\left[\begin{pmatrix} F \\ P \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix} \right] \leq \frac{1}{2}\sqrt{\frac{c_4}{c_3}} \iint \left\{ (\operatorname{grad} F)^2 + (\operatorname{rot} P)^2 \right\} dxdy + 2 \oint\limits_{\Gamma} \frac{P^2}{A(l)} \, dl. \tag{31}$$

Observe that in the two-dimensional case we have

$$(\operatorname{rot} P)^2 = (\operatorname{grad} P)^2. \tag{32}$$

To continue estimation from below, we consider the functions F and P separately. We use the following inequality for Ω^E which results from the theorem about equivalent norms of $W_2^{(1)}(\Omega)$ [7, 8]:

$$\left(\iint\limits_{\Omega^E} F^2 \, dxdy \right)^{1/2} \leq \sqrt{c_7} \left\{ \left| \oint\limits_{\Gamma} a(l) F \, dl \right| + \left(\iint\limits_{\Omega^E} (\operatorname{grad} F)^2 \, dxdy \right)^{1/2} \right\}, \tag{33}$$

where the function $a(l)$ is such that the linear functional is bounded and does not vanish for $F(x, y) \equiv 1$. Take $a(l) = 1$. By the second condition of (19), the integral over Γ vanishes and from (33) we obtain

$$\iint\limits_{\Omega^E} F^2 \, dxdy \leq c_7 \iint\limits_{\Omega^E} (\operatorname{grad} F)^2 \, dxdy. \tag{34}$$

From the embedding theorem of $W_2^{(1)}(\Omega^E)$ into $L_2(\Gamma^E)$ and (34) we derive

$$\oint\limits_{\Gamma^E} F^2 \, dl^E \leq c_8 \iint\limits_{\Omega^E} (\operatorname{grad} F)^2 \, dxdy. \tag{35}$$

Consider the integral of F over Γ^E. Estimate it using the Cauchy–Bunyakovskiĭ inequality and (35):

$$\left| \oint\limits_{\Gamma^E} F \, dl^E \right| \leq \left(\oint\limits_{\Gamma^E} F^2 \, dl^E \right)^{1/2} \left(\oint\limits_{\Gamma^E} dl^E \right)^{1/2}.$$

Since Γ^E has finite length, from this inequality and (35) we obtain

$$\left| \oint\limits_{\Gamma^E} F \, dl^E \right| \le \left(c_8 |\Gamma^E| \iint\limits_{\Omega^E} (\operatorname{grad} F)^2 \, dxdy \right)^{1/2}. \tag{36}$$

Now, we can apply (33) to F in Ω^N and Ω^S, since by (20) F preserves its value upon passage from Γ^E to Γ^N or Γ^S. We can rewrite the integral on the left-hand side of (36) as

$$\oint\limits_{\Gamma^S} F\big(l^E(l^S)\big) \frac{dl^E}{dl^S} \, dl^S. \tag{37}$$

It has the shape of the linear functional in (33) with the function

$$a(l) = \frac{dl^E}{dl^S}$$

strictly positive and bounded by virtue of (1). Therefore, for $F \equiv 1$ the integral does not vanish and hence we can use an inequality like (33) in Ω^S with another constant c_9. Since (36) holds for (37), we find that

$$\left(\iint\limits_{\Omega^S} F^2 \, dxdy \right)^{1/2} \le c_9 \left(\left(c_8 |\Gamma^E| \iint\limits_{\Omega^E} (\operatorname{grad} F)^2 \, dxdy \right)^{1/2} + \left(\iint\limits_{\Omega^S} (\operatorname{grad} F)^2 \, dxdy \right)^{1/2} \right).$$

Take the square of this inequality, gather the integrals in one, and increase it by adding the integral of a nonnegative function over Ω^N to obtain

$$\iint\limits_{\Omega^S} F^2 \, dxdy \le c_{10} \iint (\operatorname{grad} F)^2 \, dxdy, \tag{38}$$

where $c_{10} = 2c_9^2 \max(c_8|\Gamma^E|, 1)$. Therefore, for F in Ω^S we deduce an inequality similar to (34) but with integration over all subdomains on the right-hand side.

We can establish the same inequality for Ω^N with some constant c_{11}. Summing up, we arrive at the following inequality for the whole domain Ω:

$$\iint F^2 \, dxdy \le (c_7 + c_{10} + c_{11}) \iint (\operatorname{grad} F)^2 \, dxdy. \tag{39}$$

Apply to P the inequality like (33) with $a(l) = 1/A(l)$. The function $a(l)$ satisfies the necessary conditions by (9). The linear functional in the inequality of (33) vanishes on the set of functions P in question by virtue of the last equality of (19). Therefore, for P we obtain an inequality similar to (34) together with the following estimate which differs from (36) only in the value of the constant:

$$\left| \oint\limits_{\Gamma^E} P(l^E) \, dl^E \right| \le c_{12} \left(\iint\limits_{\Omega^E} (\operatorname{grad} P)^2 \, dxdy \right)^{1/2}. \tag{40}$$

Applying (22), in Ω^N we express the integral over Γ^S for Ω^S by using (21):

$$\oint\limits_{\Gamma^S} P(l^S) \frac{dl^E}{dl^S} \, dl^S = \oint\limits_{\Gamma^S} P(l^E(l^S)) \frac{dl^E}{dl^S} \, dl^S - \oint\limits_{\Gamma^S} P(l^N(l^E(l^S))) \frac{dl^E}{dl^S} \, dl^S. \tag{41}$$

Transform the integrals on the right-hand side:

$$\oint\limits_{\Gamma^E} P(l^E)\,dl^E - \oint\limits_{\Gamma^N} P(l^N)\frac{dl^E}{dl^N}\,dl^N.$$

The last integral vanishes by (22).

Using (40), we obtain an estimate for the left-hand side of (41) which enables us to apply (33) and obtain an inequality for P in Ω^S similar to (38). Combining the inequalities obtained in the three subdomains, we arrive at an inequality for P in the whole domain Ω which is similar to (39). Denote by c_{13} the least constant in the inequalities like (39) for F and P. Using (32) we can continue estimate (30):

$$\left[\begin{pmatrix} F \\ P \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix}\right] \geq \frac{1}{2c_{13}}\sqrt{\frac{c_3}{c_4}} \iint (F^2 + P^2)\,dxdy. \tag{42}$$

We have thus proven positive definiteness of the quadratic form that corresponds to the bilinear form (23). Since the latter is symmetric, we can use the quadratic form as an inner product on the set of pairs F, P of smooth functions satisfying (19)–(22).

Now, we continue estimate (31) from above for the value of the quadratic form corresponding to the bilinear form (23).

By boundedness of $A(l)$, (9), and the embedding theorem of $W_2^{(1)}(\Omega^E)$ into $L_2(\Gamma)$, we can estimate the absolute value of the last integral in (31) from above by

$$\frac{c_{14}}{c_3} \iint\limits_{\Omega^E} (\operatorname{grad} P)^2\,dxdy.$$

In view of (32), we can include this quantity in the first term on the right-hand side of (31) by changing the value of the constant:

$$\left[\begin{pmatrix} F \\ P \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix}\right] \leq \left(\frac{1}{2}\sqrt{\frac{c_4}{c_3}} + 2\frac{c_{14}}{c_3}\right) \iint \left\{(\operatorname{grad} F)^2 + (\operatorname{rot} P)^2\right\} dxdy. \tag{43}$$

Inequalities (30), (42), and (43) mean the equivalence of the energy norm and the sum of the norms of F and P as elements of $W_2^{(1)}(\Omega)$.

6. Minimum of the energy functional. By the energy functional we mean

$$W(F,P) = \frac{1}{2}\left[\begin{pmatrix} F \\ P \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix}\right] - \iint (FQ + PG)\,dxdy. \tag{44}$$

We suppose that the given functions Q and G have finite norms as elements of $L^2(\Omega)$. Therefore, the linear part of the energy functional is a bounded linear functional and, by Riesz's theorem, it is representable as the inner product with some element of the energy space:

$$\iint (FQ + PG)\,dxdy = \left[\begin{pmatrix} F_0 \\ P_0 \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix}\right].$$

Since the quadratic part of $W(F,P)$ is positive definite, a minimum exists and is unique. The minimum is attained at $(F,P) = (F_0, P_0)$.

7. Weak solution. Minimality of $W(F,P)$ means that

$$\frac{d}{dt}W(F+tu, P+tv)\bigg|_{t=0} = 0$$

for arbitrary functions u and v in the energy space. According to (44), this implies

$$\left[\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} F \\ P \end{pmatrix}\right] - \iint (uQ + vG)\, dxdy = 0.$$

Consider this minimality condition for $W(F,P)$ under the additional assumption that the functions F and P are smooth. In the bilinear form defined by (23), we then can integrate by parts in each of the three subdomains. We use the abbreviations $\mathbf{E}$, (16), and $\mathbf{J} = \sigma\mathbf{E}$, (3):

$$\iint \{u(\operatorname{div}\mathbf{J} - Q) + v(\operatorname{rot}_z \mathbf{E} - G)\}\, dxdy + \oint_{\Gamma} (-uJ_n - vE_l)\, dl$$
$$+ \oint_{\Gamma^E} (-uJ_n + vE_l)\, dl^E + \oint_{\Gamma^N} (-uJ_n - vE_l)\, dl^N + \oint_{\Gamma^S} (-uJ_n - vE_l)\, dl^S = 0. \tag{45}$$

The opposite sign of vE_l in the integral over Γ^E relates to the choice of the positive direction on Γ^E.

By the arbitrariness of u and v inside the subdomains, from (45) we routinely deduce the equalities

$$\operatorname{div}\mathbf{J} = Q, \quad \operatorname{rot}_z \mathbf{E} = G. \tag{46}$$

Therefore, (45) reduces to the vanishing of the sum of the boundary integrals. Since u and v on the exterior boundary Γ are not connected with their values on the interior boundary, we have one more identity

$$\oint_{\Gamma} (uJ_n + vE_l)\, dl = 0 \tag{47}$$

which can be written as follows on using (19), or (26) equivalent to the former in the case of smooth functions:

$$\oint_{\Gamma} \left(uJ_n + A(l)\frac{\partial u}{\partial l} E_l\right)\, dl = 0.$$

We can integrate the second term by parts and obtain

$$\oint_{\Gamma} u\left(J_n - \frac{\partial}{\partial l}(A(l)E_l)\right)\, dl = 0. \tag{48}$$

We collect the remaining three integrals over the interior boundary in (45) by expressing the three functions $u(l^N)$, $u(l^S)$, and $v(l^E)$ in terms of $u(l^E)$, $v(l^N)$, and $v(l^S)$ by means of (20) and (21) valid for u and v in the energy space. The last three functions are arbitrary; only the mean of $v(l^N)$ is fixed by (22). Thus, from (45) we obtain three more independent identities:

$$\oint_{\Gamma^E} u(l^E)\left\{J_n(l^E) + \frac{dl^N}{dl^E}J_n(l^N) + \frac{dl^S}{dl^E}J_n(l^S)\right\}\, dl^E = 0, \tag{49}$$

$$\oint_{\Gamma^N} v(l^N)\left\{-E_l(l^N) + \frac{dl^E}{dl^N}E_l(l^E)\right\}\, dl^N = 0, \tag{50}$$

$$\oint_{\Gamma^S} v(l^S)\left\{-E_l(l^S) + \frac{dl^E}{dl^S}E_l(l^E)\right\}\, dl^S = 0. \tag{51}$$

Since the function $u(l^E)$ is arbitrary, (49) implies validity of the conjugation condition (5).

By the arbitrariness of $v(l^S)$, from (51) we deduce the second condition of (6).

We return to the first condition in (6) later. Now, we turn to (48).

The function $u(l)$ on Γ has zero mean in view of the second condition of (19). Denote by $\tilde{u}(l)$ the boundary value of a function $\tilde{u}(x, y)$ which, unlike $u(x, y)$, may fail to satisfy the second condition of (19). According to condition (26) equivalent to (19), we construct the function $v(l)$ on Γ as follows:

$$v(l) = A(l)\frac{\partial \tilde{u}(l)}{\partial l},$$

and extend it smoothly to Ω^E so that $v(x, y)$ vanishes on Γ^E. Then we can extend $v \equiv 0$ to Ω^N and Ω^S. For every such pair $\tilde{u}$, v we can construct a pair $u = \tilde{u} - u_0$, v which satisfies all conditions (19)–(22); to this end, we take the constant u_0 to be

$$u_0 = \frac{1}{|\Gamma|}\oint_\Gamma \tilde{u}\, dl.$$

Insert $u(l) = \tilde{u}(l) - u_0$ in the identity (48) under consideration:

$$\oint_\Gamma \tilde{u}\left(J_n - \frac{\partial}{\partial l}(A(l)E_l)\right) dl - u_0 \oint_\Gamma J_n\, dl + u_0 \oint_\Gamma \frac{\partial}{\partial l}(A(l)E_l)\, dl = 0. \tag{52}$$

The last integral vanishes identically, since the curve Γ is closed. Integrate the above-proven first equality of (46) separately over the subdomains Ω^E, Ω^N, and Ω^S and then apply the Gauss–Ostrogradskiĭ formula to the left-hand sides:

$$\oint_\Gamma J_n\, dl + \oint_{\Gamma^E} J_n\, dl^E = \iint_{\Omega^E} Q\, dxdy,$$

$$\oint_{\Gamma^N} J_n(l^N)\, dl^N = \iint_{\Omega^N} Q\, dxdy, \quad \oint_{\Gamma^S} J_n(l^S)\, dl^S = \iint_{\Omega^S} Q\, dxdy.$$

Validity of (5) is already proven. Therefore, summing up these equalities, we obtain

$$\oint_\Gamma J_n\, dl = \iint Q\, dxdy.$$

The right-hand side vanishes by (7). Therefore, the factor of u_0 in (52) vanishes. The function $\tilde{u}(l)$ in the identity resulting from (52) is arbitrary; therefore, its factor equals zero; i.e., the boundary condition (4) is valid.

Now, consider identity (50). Denote by $\tilde{v}(l^N)$ the boundary value of an arbitrary function $\tilde{v}(x, y)$ which, unlike $v(x, y)$, may fail to satisfy condition (22). Here we are interested only in the functions $\tilde{v}$ vanishing inside Ω^S and on the exterior boundary Γ and in the identically zero functions u. In this case condition (21) becomes equivalent to $\tilde{v}(l^E) = \tilde{v}(l^N)$. This pairs of functions satisfy the principal boundary conditions (19)–(21). For each function $\tilde{v}(x, y)$ we can construct a function $v(x, y)$ by subtracting a constant v_0 from $\tilde{v}(x, y)$ in Ω^N:

$$v_0 = \frac{1}{|\Gamma^E|}\oint_{\Gamma^N} \tilde{v}(l^N)\frac{dl^E}{dl^N}\, dl^N,$$

and subtracting from $\tilde{v}(x, y)$ in Ω^N some function that results from interpolating v_0 on Γ^E and zero on Γ into Ω^E. This pair u, v satisfies all principal boundary conditions (19)–(22). Inserting these expressions in (50), we obtain

$$\oint_{\Gamma^N} \tilde{v}(l^N)\left\{-E_l(l^N) + \frac{dl^E}{dl^N}E_l(l^E)\right\} dl^N - v_0 \oint_{\Gamma^N} E_l(l^N)\, dl^N + v_0 \oint_{\Gamma^E} E_l(l^E)\, dl^E = 0. \tag{53}$$

Integrate the above-proven second equality of (46) separately over Ω^N and Ω^S and then apply the Gauss–Ostrogradskiĭ formula to the left-hand sides:

$$\oint\limits_{\Gamma^N} E_l(l^N)\, dl^N = \iint\limits_{\Omega^N} G\, dxdy, \quad \oint\limits_{\Gamma^S} E_l(l^S)\, dl^S = \iint\limits_{\Omega^S} G\, dxdy.$$

Subtracting these equalities and using the second of the above-proven equality (6), we find that

$$\oint\limits_{\Gamma^N} E_l(l^N)\, dl^N - \oint\limits_{\Gamma^E} E_l(l^E)\, dl^E = \iint\limits_{\Omega^N} G\, dxdy - \iint\limits_{\Omega^S} G\, dxdy.$$

The right-hand side vanishes by condition (7) on the right-hand sides which is necessary for solvability of the problem. We obtain

$$\oint\limits_{\Gamma^E} E_l(l^E)\, dl^E - \oint\limits_{\Gamma^N} E_l(l^N)\, dl^N = 0.$$

Therefore, the factor of v_0 in (53) vanishes and, by the arbitrariness of $\tilde{v}(l^N)$, we obtain

$$E_l(l^E) = \frac{dl^N}{dl^E} E_l(l^N);$$

i.e., we have proven the first equality of (6). Thus, we completed validation of all boundary conditions (4)–(6) that result from minimizing the energy functional. These boundary conditions are called natural, as opposed to the principal boundary conditions (19)–(22).

Thus, the functions F and P providing a minimum to the energy functional enable us to construct, using (16) and (3), vector functions $\mathbf{E}$ and $\mathbf{J}$ that are solutions to the original boundary value problem (2)–(6), provided that F and P are additionally assumed smooth. By Lemma 1, such a solution is unique. We can routinely [3] prove the converse assertion: a solution to the boundary value problem provides a minimum to the energy functional.

In the general case, a pair F, P of functions providing a minimum to the energy functional has the finite energy norm equivalent to the $W_2^{(1)}(\Omega)$ norm. We call such a pair F, P a weak solution to problem (2)–(6), (19)–(22), where $\mathbf{E}$ and $\mathbf{J}$ are considered as the notations for (16) and (3).

By (16), the equations (2) and (3) for F and P themselves have the shape (18). We can rewrite the operator of (18) as the product of first-order operators:

$$\begin{pmatrix} \operatorname{div} \sigma \\ \operatorname{rot}_z \end{pmatrix} S(\sigma^T \operatorname{grad}, \operatorname{rot}) \begin{pmatrix} F \\ P \end{pmatrix} = \begin{pmatrix} Q \\ G \end{pmatrix}.$$

This operator has the shape of $L^T SL$, since the operators div and rot_z are adjoint to grad and rot while the matrix S is symmetric. The operators of this form are called adjointly factorized. This property enables us to simplify our approach to grid models [9] essentially.

The existence and uniqueness of a weak solution follows from the result of Section 6 about the existence and uniqueness of an element of the energy space which minimizes the value of the energy functional. The functions $\mathbf{J}$ and $\mathbf{E}$ have finite $L_2(\Omega)$ norms and constitute a weak solution to (2)–(6).

8. Thermodynamics. For S like (17) we can rewrite the quadratic form (23), using (16) and (3), as

$$\iint \mathbf{E}^T \mathbf{J}\, dxdy.$$

The integrand is the Joule dissipation density; i.e., the integral equals the thermal energy produced in a conductor in unit time due to the electric current flux. Therefore, this inner product is called the energy inner product. Considering

$$S^{-1} = (\sigma + \sigma^T)/(2T) \tag{54}$$

in place of (17), the energy quadratic form equals

$$\iint \frac{1}{T}\mathbf{E}^T\mathbf{J}\, dxdy,$$

i.e., the entropy production rate for a given distribution of the absolute temperature $T(x, y)$. The specific shape of the matrix S was used only in estimation of the interval (29) containing the eigenvalues of the matrix (28). In [2], using a special choice of the coefficient with which the auxiliary integral (24) is added to the quadratic form (23), the author obtained a similar estimate for an arbitrary symmetric and uniformly (in Ω) positive definite S. Positivity of the absolute temperature and the conditions on σ, (8), guarantee this property for S given by (54).

Thus, from the viewpoint of the nonequilibruim thermodynamics the square of the energy norm has the meaning of the entropy production rate.

9. Other boundary value problems. We can impose simpler conditions on the exterior boundary in comparison with (4):

$$J_n|_\Gamma = 0 \tag{55}$$

if an ideal isolator is beyond the boundary or

$$E_l|_\Gamma = 0 \tag{56}$$

if an ideal conductor is beyond the boundary. It is natural to call these problems the first and second boundary value problems as in the case of a schlicht domain [3]. We now list the changes to be made in the statements and proofs given above for the boundary condition (4).

The conjugation conditions (5) and (6) on the interior boundary remain the same. The necessary constraints (7) on the right-hand sides for solvability of the problem remain the same for the first problem and become

$$\iint_{\Omega^N} G\, dxdy + \iint_{\Omega^E} G\, dxdy = 0, \quad \iint_{\Omega^N} G\, dxdy = \iint_{\Omega^S} G\, dxdy \tag{57}$$

for the second problem.

The proofs of uniqueness of a solution to the problems become slightly simpler, since integral (15) has the zero integrand for either of the conditions (55) and (56).

The principal boundary conditions (20)–(22) on the interior boundary remain the same. In the first boundary value problem we replace condition (19) on the exterior boundary with

$$P|_\Gamma = 0, \quad \oint_\Gamma F\, dl = 0, \tag{58}$$

and in the second, with

$$F|_\Gamma = 0, \quad \oint_\Gamma P\, dl = 0. \tag{59}$$

Observe that the vanishing of the mean values in (58) and (59) serves for elimination of arbitrary additive constants to within which the functions F and P are actually determined. As it is usual for the Neumann problem, we could fix the mean value of the function F or P over the domain; however, in this case we would need Poincaré's inequality in a nonschlicht domain. Using an inequality like (33) seems to be simpler, especially as the linear functional with the function $a(l) = 1$ in it takes the zero value for both functions F and P satisfying (58) or (59). Note that we could use Friedrichs' inequality instead of (33) for P in the first problem and for F in the second.

Since the last integral in (25) vanishes for either of the conditions (58) and (59), the auxiliary integral (24) vanishes. Therefore, the boundary integral in the upper estimate (31) for the quadratic form goes away and there is no need to estimate it for obtaining (43), wherein we would have $c_{14} = 0$.

Studying a weak solution, we only modify the analysis of identity (47). Validity of (55) or (56) is deduced from this identity due to the arbitrariness of F to within (58) or P to within (59) on this boundary. We avoid fixing the mean values of these functions over Γ in the same way by adding an arbitrary constant whose coefficient, as in (52), turns out to be zero in view of the conditions (7) or (57) on the right-hand sides.

Thus, we establish that solving the first (55) or the second (56) boundary value problems is equivalent to minimizing the energy functional (44) in the corresponding energy space.

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